Some Theorems and Proofs Regarding Orthogonal Projections*

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1 Theorems

Theorem 1. *(Theorem 2.3, part 1 in [3]) For any matrix A and B the following statement is true.*

Let P_A be orthogonal projector onto the range of A and let P_B be orthogonal projector onto the range of B. If $rank(A) = rank(B)$:

1. *The singular values of $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ are the same.*
2. *The nonzero singular values σ_i of $P_A P_B^{\perp}$ correspond to pairs $\pm\sigma_i$ of eigenvalues of $P_B - P_A$, so that*

$$\|P_B - P_A\|_2 = \|P_A P_B^{\perp}\|_2 = \|P_B P_A^{\perp}\|_2 \tag{1}$$

Theorem 2. *(Theorem 2.3, part 2 in [3]) For any two matrices $A_{m\times n}$ and $B_{m\times l}$ and two orthogonal projectors P_A and P_B onto the ranges of A and B respectively we have:*

$$\|P_B - P_A\|_2 < 1 \Rightarrow rank(A) = rank(B) \tag{2}$$

1.1 Auxiliary Proofs

Lemma 1. *Let X and Y be subspaces of $\mathbb{C}^n$ and P_X, P_Y orthogonal projections onto X and Y, respectively. σ_i is a singular value of $P_X P_Y$ if and only if σ_i^2 is an eigenvalue of $P_X P_Y$.*

*This work resulted from me reading [3] and trying to understand all the steps. The theorems stated here are stated but not proven in [3]. Instead, references are cited for the proofs. I went through some of the references and presented the proofs here. Some proofs follow the proofs in the references closely, some rely on the knowledge I gained from the references and adapted into my own proofs.

Proof. This proof was taken from [5] and slightly expanded.

We are going to prove the lemma in two parts. First, we will prove that if σ_i is a singular value of $P_X P_Y$ then $\lambda_i = \sigma_i^2$ is an eigenvalue of $P_X P_Y$. Then we shall prove that for each eigenvalue λ_i of $P_X P_Y$ there is a singular value $\sigma_i = \sqrt{\lambda_i}$ of $P_X P_Y$.

- **First direction**

  We know that singular values of a matrix A are square roots of eigenvalues of matrix A^*A or AA^*. In other words:

  $$P_X P_Y (P_X P_Y)^* v = \sigma_i^2 \cdot v \tag{3}$$

  for some eigenvector v of $P_X P_Y (P_X P_Y)^*$.

  We have:

  $$P_X P_Y (P_X P_Y)^* v = P_X P_Y P_Y^* P_X^* v = P_X P_Y P_X v = \sigma_i^2 v \tag{4}$$

  From expression above we see that, **if $\boldsymbol{\sigma_i > 0}$** v is in the range of P_X which means:

  $$P_X P_Y v = \sigma_i^2 v \tag{5}$$

  This shows that for every non-zero singular value σ_i of $P_X P_Y$ there exists an eigenvalue σ_i^2 of $P_X P_Y$.

  Now to complete this direction of the proof we shall show that for every 0 valued singular value of $P_X P_Y$ there is a 0 valued eigenvalue of that same matrix. We know that in order for $P_X P_Y$ to have a 0 valued singular value that there has to exist a vector v such that $(P_X P_Y)^* P_X P_Y v = 0$. If we multiply the equality with v^* on the left we get: $v^*(P_X P_Y)^* P_X P_Y v = \|P_X P_X v\|_2 = 0 \rightarrow P_X P_Y v = 0$ which means that v is an eigenvector of $P_X P_Y$ with eigenvalue 0. Hence, if 0 is a singular value of $P_X P_Y$ it is also its eigenvalue.

- **Second direction** We start by showing that for each eigenvalue $\lambda_i > 0$ of $P_X P_Y$ there is a singular value $\sigma_i = \sqrt{\lambda_i}$ of the same matrix:

  $$P_X P_Y v = \lambda_i \cdot v \tag{6}$$

  $$P_X P_Y v = P_X P_Y P_X v = P_X P_Y P_Y^* P_X^* v = P_X P_Y (P_X P_Y)^* v = \lambda_i \cdot v \tag{7}$$

  Hence, every eigenvector-non zero eigenvalue pair of $P_X P_Y$ is also an eigenvector-eigenvalue pair of $P_X P_Y (P_X P_Y)^*$. This means that there is a singular value $\sigma_i = \sqrt{\lambda_i}$. In equation (7) we could substitute v with $P_X v$ because for $\lambda_i \neq 0$, v is in range of P_X.

Next we prove that if 0 is an eigenvalue of $P_X P_Y$ that it is also an eigenvalue of $(P_X P_Y)^* P_X P_Y$. By our assumption we have $P_X P_Y v = 0$. This entails $(P_X P_Y)^* P_X P_Y v = (P_X P_Y)^* \cdot 0 = 0$ which means that if $P_X P_Y$ has an eigenvalue $\lambda_i = 0$ then so does $(P_X P_Y)^* P_X P_Y$. This ends the proof of the second direction.

□

Corollary 1.1. *Let P_X and P_Y be orthogonal projectors. For each eigenvalue λ_i of $P_X P_Y$ we have $0 \leq \lambda_i \leq 1$. Same is true for each singular value of σ_i of $P_X P_Y$, $0 \leq \sigma_i \leq 1$.*

Proof. Lemma 1 states that each eigenvalue λ_i of $P_X P_Y$ corresponds to the square of a singular value of $P_X P_Y$: $\lambda_i = \sigma_i^2 \geq 0$ which proves the lower bound. We can use the fact that for orthogonal projectors $\|P_X\|_2 = \|P_Y\|_2 = 1$ to prove the upper bound:

$$\lambda_i = |\lambda_i| = \|\lambda_i v\|_2 / \|v\|_2 = \|P_X P_Y v\|_2 / \|v\|_2 \leq \|P_X\|_2 \|P_Y\|_2 \|v\|_2 / \|v\|_2 = 1 \quad (8)$$

Thus we have proven $0 \leq \lambda_i \leq 1$. From (8) we get $\sigma_i = \sqrt{\lambda_i} \leq 1$ and since singular values are positive by definition we have $0 \leq \sigma_i \leq 1$

□

Next, the concept of reciprocal vectors is introduced and some facts about the reciprocal vectors are stated and proven. This part is taken from [1].

Definition 1.1. Reciprocal vectors Let us observe two orthogonal projectors P_A and P_B. Let vector u be in $range(P_A)$ and vector v be in $range(P_B)$. We say that vectors u and v are reciprocal to each other and call them a pair of reciprocal vectors if the following holds: $\alpha \cdot u = P_A v$ and $\beta \cdot v = P_B u$ for some scalars $\alpha \neq 0, \beta \neq 0$. In other words, vector u is projected into the ray that spans vector v and vice versa.

Lemma 2. *Any reciprocal vectors u, v in the range of P_A and P_B are eigenvectors of $P_A P_B$ and $P_B P_A$ respectively with eigenvalue $\cos^2\theta$ where θ represents the angle between u and v and $\cos\theta = u^* v / (\|u\|_2 \|v\|_2)$. Also, if u is an eigenvector of $P_A P_B$ for a nonzero eigenvalue $\cos^2\theta$ then $v = P_B u$ is an eigenvector of $P_B P_A$ for the same eigenvalue and u and v are reciprocal vectors in column spaces of P_A and P_B respectively with angle θ between them.*

Proof. Based on the definition of reciprocal vectors we can write:

$$P_A P_B u = P_A \beta v = \beta P_A v = \beta\alpha \cdot u \quad (9)$$

Similarly we have:

$$P_B P_A v = \beta\alpha \cdot v \quad (10)$$

We see that each pair of reciprocal vectors corresponds to the eigenvectors of matrices $P_A P_B$ and $P_B P_A$ respectively with corresponding eigenvalue for both eigenvectors being the same: $\alpha \cdot \beta$.

We can express the eigenvalue in terms of the angle between the two reciprocal vectors:

$$\alpha\|u\|_2^2 = u^*u\alpha = u^*P_Av = u^*P_A^*v = (P_Au)^*v = u^*v \tag{11}$$

$$\beta\|v\|_2^2 = v^*v\beta = v^*P_Bu = v^*P_B^*u = (P_Bv)^*u = v^*u \tag{12}$$

$$\alpha \cdot \beta = (u^*v)^2/(\|u\|_2^2\|v\|_2^2) = cos^2\theta \tag{13}$$

θ being the angle between the two vectors i.e. the rays they span.

From the equations (9)-(13) we see that each pair of reciprocal vectors corresponds to eigenvectors of P_AP_B and P_BP_A with eigenvalue $cos^2\theta$.

Now we want to prove the second part of the Lemma: that each eigenvector u of P_AP_B with nonzero corresponding eigenvalue makes a reciprocal pair of vectors with an eigenvector v of P_BP_A with the same nonzero eigenvalue. Let u be an eigenvector of P_AP_B with eigenvalue λ_i: $P_AP_Bu = \lambda_iu$. Let v be $v = P_Bu$. Then we have $P_Av = P_AP_Bu = \lambda_iu$ and $P_BP_Av = P_B\lambda_iu = \lambda_iv$ meaning that v is an eigenvector of P_BP_A with same eigenvalue and u and v are reciprocal vectors with $\alpha = \lambda_i$ and $\beta = 1$. The angle between the two vectors is by (13) $\theta = \arccos\sqrt{\alpha\beta} = \arccos\sqrt{\lambda_i}$ and thus the eigenvalue $\lambda_i = cos^2\theta$.

To sum up, each eigenvalue $\lambda_i > 0$ of P_AP_B and P_BP_A is attached to a pair of reciprocal vectors for P_A and P_B. □

Lemma 3. *Let P_X and P_Y be orthogonal projectors onto some subspaces X and Y of $\mathbb{C}^n$. $0 < \lambda_i' < 1$ is an eigenvalue of P_XP_Y and P_YP_X if and only if $\lambda_i = 1 - \lambda_i'$ is an eigenvalue of both $P_XP_Y^\perp$ and $P_YP_X^\perp$. Note: the condition $0 \leq \lambda_i' \leq 1$ can be restated as $\lambda_i' \neq 0 \land \lambda_i' \neq 1$ because the interval for possible eigenvalues for a product of orthogonal projectors is $[0, 1]$ (see Corollary 1.1).*

Proof. Let us observe the following for some eigenvalue $0 < \lambda_i < 1$ of $P_XP_Y^\perp$:

$$P_XP_Y^\perp \cdot v = \lambda_i \cdot v \tag{14}$$

$$P_X(I - P_Y)v = P_Xv - P_XP_Yv = \lambda_i \cdot v \tag{15}$$

Since $\lambda_i > 0$ we know that $P_Xv = v$ so based on equation (15) we can write:

$$P_XP_Yv = (1 - \lambda_i)v \tag{16}$$

We see that v is an eigenvector of P_XP_Y with eigenvalue $1 - \lambda_i$. Since $(P_XP_Y)^* = P_YP_X$ matrices P_XP_Y and P_YP_X have the same eigenvalues. In other words there is an eigenvector u for matrix P_YP_X such that:

$$P_YP_Xu = (1 - \lambda_i)u \Rightarrow u - P_YP_Xu = \lambda_iu \tag{17}$$

Since we are observing $0 < \lambda_i < 1$, we have $0 < 1 - \lambda_i < 1$ which means $P_Yu = u$ so we can write:

$$u - P_Y P_X u = P_Y u - P_Y P_X u = P_Y (I - P_X) u = P_Y P_X^{\perp} = \lambda_i u \quad (18)$$

We have thus proven that any eigenvalue $0 < \lambda_i < 1$ of $P_X P_Y^{\perp}$ is also an eigenvalue of $P_Y P_X^{\perp}$ and there exists an eigenvalue $\lambda_i' = 1 - \lambda_i$ of both $P_X P_Y$ and $P_Y P_X$. Similarly, we can start with an eigenvalue λ_i of $P_Y P_X^{\perp}$ and prove that $P_X P_Y^{\perp}$ has the same eigenvalue and that both $P_X P_Y$ and $P_Y P_X$ have an eigenvalue $\lambda_i' = 1 - \lambda_i$.

Now we need to prove the other direction of our lemma: that for each $0 < \lambda_i' < 1$ eigenvalue of both $P_X P_Y$ and $P_Y P_X$ there is an eigenvalue $\lambda_i = 1 - \lambda_i'$ of both $P_X P_Y^{\perp}$ and $P_Y P_X^{\perp}$:

$$P_X P_Y v = \lambda_i' v \Rightarrow P_X P_Y v - P_X v = \lambda_i' v - v \Rightarrow P_X P_Y^{\perp} v = (1 - \lambda_i') v \quad (19)$$

Similarly we can get $P_Y P_X^{\perp} u = (1 - \lambda_i') u$ where u is an eigenvector of $P_Y P_X$ associated with the eigenvalue λ_i'. We have thus proven that $P_X P_Y^{\perp}$ and $P_Y P_X^{\perp}$ have an eigenvalue $\lambda_i = 1 - \lambda_i'$, $0 < \lambda_i < 1$, for each eigenvalue $0 < \lambda_i' < 1$ of $P_X P_Y$ and $P_Y P_X$. □

1.2 Theorem Proofs

Proof of Theorem 1, Part 1. Since $P_A^{\perp}$ and $P_B^{\perp}$ as well as P_A and P_B are all orthogonal projectors by Lemma 1 we can compare eigenvalues of $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ in lieu of their singular values.

We did a lion's share of work by proving Lemma 3. By Lemma 3, we know that $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ share all eigenvalues $0 < \lambda_i < 1$. By Lemma 1 we know that applies to the singular values $0 < \sigma_i < 1$ of $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ as well.

Since both $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ are products of orthogonal projectors, their singular values are in range $0 \leq \sigma_i \leq 1$ (see Corollary 1.1). We have proven that $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ share all the singular values in the range $0 < \sigma_i < 1$. What is left to prove is that both matrices have the same number of singular values with value 1. We can analyze the trace of matrices $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ to prove this.

We know that the trace of a matrix is equal to the sum of its eigenvalues.[1] We also know that all the eigenvalues of any orthogonal projector P are either 0 or 1. We can verify that by observing the singular value decomposition of an orthogonal projector (see [4], proof of Theorem 6.1) which is also its eigenvalue decomposition. These two facts imply $trace(P) = rank(P)$ for any orthogonal projector P.

[1]This can be derived from the fact that each matrix can be written in the Jordan canonical form (see [2]) $A = SJS^{-1}$ where J has eigenvalues on its diagonal and the fact that $trace(AB) = trace(BA)$. The equality $trace(AB) = trace(BA)$ can be proven by simply writing out the sums that correspond to the traces of matrices AB and BA and rearranging them to be in the same form. By using the equality $trace(AB) = trace(BA)$ we can write $trace(A) = trace(SJS^{-1}) = trace(S^{-1}SJ) = trace(J) = \sum_i \lambda_i$ where λ_i are eigenvalues of A. For orthogonal projectors we could use the eigenvalue decomposition to prove the fact that $trace(P) = \sum_i \lambda_i$. By using the Jordan canonical form we get a more general result.

We can write:

$$\begin{aligned} rank(A) = trace(P_A) &= trace(P_A(P_B + P_B^{\perp})) \\ &= trace(P_A P_B) - trace(P_A P_B^{\perp}) \end{aligned} \tag{20}$$

$$\begin{aligned} rank(B) = trace(P_B) &= trace(P_B(P_A + P_A^{\perp})) \\ &= trace(P_B P_A) - trace(P_B P_A^{\perp}) \end{aligned} \tag{21}$$

We know that $(P_A P_B)^* = P_B P_A \Rightarrow trace(P_A P_B) = trace(P_B P_A)$. Theorem 1. assumes $rank(A) = rank(B)$. Based on this and the equations (20) and (21) we get:

$$trace(P_A P_B^{\perp}) = trace(P_B P_A^{\perp}) \tag{22}$$

Let us delve deeper into equation (22). We denote the number of eigenvalues with value 1 of matrix $P_A P_B^{\perp}$ as $n_{AB^{\perp}}$ and for matrix $P_B P_A^{\perp}$ as $n_{BA^{\perp}}$. We can write:

$$trace(P_A P_B^{\perp}) = \sum\nolimits_{\lambda_i \neq 1} \lambda_i(P_A P_B^{\perp}) + n_{AB^{\perp}} \cdot 1 \tag{23}$$

$$trace(P_B P_A^{\perp}) = \sum\nolimits_{\lambda_i \neq 1} \lambda_i(P_B P_A^{\perp}) + n_{BA^{\perp}} \cdot 1 \tag{24}$$

We know that $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ have the same eigenvalues in range $0 < \lambda_i < 1$, hence $\sum_{\lambda_i \neq 1} \lambda_i(P_A P_B^{\perp}) = \sum_{\lambda_i \neq 1} \lambda_i(P_B P_A^{\perp})$. From this we get $n_{AB^{\perp}} = n_{BA^{\perp}}$, i.e. $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ have exactly the same number of eigenvalues with value 1 and therefore by Lemma 1 also the same number of singular values with value 1. Hence, all the singular values of $P_A P_B^{\perp}$ and $P_B P_A^{\perp}$ are the same.

□

Proof of Theorem 1, Part 2. The second part of our theorem states that all nonzero eigenvalues of $P_A - P_B$ $\pm\sigma_i$ correspond to singular values σ_i of $P_A P_B^T$ and $P_B P_A^T$ and vice versa.

This proof is split in two parts. The first part is about proving the statement for eigenvalues $\lambda_i \neq \pm 1$ of $P_B - P_A$ and the second part focuses on the case when $\lambda_i = \pm 1$.

- $|\lambda_i| \neq 1$

  We prove this part by showing that a vector v is an eigenvector of $P_A - P_B$ with eigenvalues satisfying this condition if and only if v is a difference of reciprocal vectors of P_A and P_B. Then we give an expression for the associated eigenvalue.

  Let us try to construct eigenvectors of $P_A - P_B$ from reciprocal vectors of P_A and P_B.

We know from the Definition 1.1 of the reciprocal vectors u and v that $\alpha u = P_A v$ and $\beta v = P_B u$. Let us see whether we can tune α and β so that the difference of two reciprocal vectors $u - v$ is an eigenvector of $P_A - P_B$.

$$(P_A - P_B) \cdot (u - v) = u - \alpha u - \beta v + v = (1 - \alpha)u - (\beta - 1)v \tag{25}$$

In order for $u - v$ to be an eigenvector of $P_A - P_B$ we need to find α and β such that the following holds:

$$1 - \alpha = \beta - 1 = \lambda_i \tag{26}$$

where λ_i would be the eigenvalue associated with the eigenvector $u - v$.

From Lemma 2 we know that there is an nonzero eigenvalue λ_i' of $P_A P_B$ with value $\lambda_i' = \alpha\beta = cos^2\theta$.

In order for $u-v$ to be an eigenvector of $P_A - P_B$ we need to have $\alpha = 2-\beta$ (see (26)). We get:

$$\alpha\beta = (2 - \beta)\beta = \lambda_i' \tag{27}$$

$$\beta^2 - 2\beta + \lambda_i' = 0 \tag{28}$$

$$\beta = \frac{(2 \pm \sqrt{4 - 4\lambda_i'})}{2} = 1 \pm \sqrt{1 - \lambda_i'} \tag{29}$$

From there we have:

$$\lambda_i = \beta - 1 = \pm\sqrt{1 - \lambda_i'} \tag{30}$$

From Lemma 3 and the fact that $\lambda_i' = 1 - \lambda_i^2 \neq 0, 1$ because we are looking at $\lambda_i \neq 0, 1$ we know that there is an eigenvalue λ_i" of both $P_A P_B^\perp$ and $P_B P_A^\perp$ such that $\lambda_i" = 1 - \lambda_i'$. Using that and Lemma 1 we can write:

$$\lambda_i = \pm\sqrt{\lambda_i"} = \pm\sigma_i \tag{31}$$

with σ_i being any nonzero singular value of $P_A P_B^\perp$ and $P_B P_A^\perp$ such that $\sigma_i \neq 1$.

To complete this part of the proof, we need to prove that each nonzero eigenvalue λ_i of $P_A - P_B$ s.t. $|\lambda_i| \neq 1$ is associated with an eigenvector that is a difference of two reciprocal vectors of P_A and P_B. This is necessary to establish that $P_A - P_B$ does not have any additional eigenvalues $\lambda_i \neq \pm 1$ that are not equal to $\pm\sigma_i$ for some singular value σ_i of $P_A P_B^\perp$ and $P_B P_A^\perp$.

For each eigenvector v of $P_A - P_B$ with eigenvalue λ_i, s.t. $\lambda_i \neq 0$ we can write $P_A v - P_B v = \lambda_i v$. From there we can get $v = P_A v/\lambda_i - P_B v/\lambda_i$. Hence, if $P_A v$ and $P_B v$ are reciprocal vectors, v is a difference of two reciprocal vectors of P_A and P_B. We now prove that $P_A v$ and $P_B v$ are in fact reciprocal vectors when $|\lambda_i| \neq 1$.

We need to find $\alpha, \beta > 0$ such that $\alpha P_A v = P_A P_B v$ and $\beta P_B v = P_B P_A v$. We can find such α and β by multiplying $P_A v - P_B v = \lambda_i v$ by P_A or P_B respectively. In case of P_A we get:

$$(1 - \lambda_i) \cdot P_A v = P_A P_B v \tag{32}$$

When multiplying by P_B we get:

$$(1 + \lambda_i) \cdot P_B v = P_B P_A v \tag{33}$$

We have found $\alpha = 1 - \lambda_i$ and $\beta = 1 + \lambda_i$ which are both nonzero when $|\lambda_i| \neq 1$ which proves that $P_A v$ and $P_B v$ are reciprocal vectors for any eigenvector v of $P_A - P_B$ that is associated with an eigenvalue $|\lambda_i| \neq 1$. Hence, all eigenvectors of $P_A - P_B$ with nonzero eigenvalues $|\lambda_i| \neq 1$ are a difference of two reciprocal vectors.

- $\lambda_i = \pm 1$

  Next we want to prove that $P_A - P_B$ has eigenvalues with value ± 1 if and only if $P_A P_B^\perp$ and $P_B P_A^\perp$ have singular values $\sigma_i = 1$.

  First we prove that if $P_A P_B^\perp$ and $P_B P_A^\perp$ have eigenvalues 1 that $P_A - P_B$ has eigenvalues ± 1.

$$P_A P_B^\perp v = v \Rightarrow \|P_A P_B^\perp v\|_2 = \|v\|_2 \tag{34}$$

$$\|v\|_2 = \|P_A P_B^\perp v\|_2 \leq \|P_A\|_2 \|P_B^\perp v\|_2 \leq \|P_B^\perp v\|_2 \tag{35}$$

$$\|P_B^\perp v\|_2 \leq \|P_B^\perp\|_2 \|v\|_2 = \|v\|_2 \tag{36}$$

$$\|v\|_2 \leq \|P_B v^\perp\|_2 \leq \|v\|_2 \Rightarrow \|P_B^\perp v\|_2 = \|v\|_2 \tag{37}$$

  Since $P_B^\perp$ is an orthogonal projector we know that for any vector v we have $\|P_B v\|_2^2 + \|P_B^\perp v\|_2^2 = \|v\|_2^2$. From there and equation (37) we get $\|P_B v\|_2 = 0 \Rightarrow P_B v = 0 \wedge P_B^\perp v = v$. This reduces the eigenvector-eigenvalue expression from equation (34) to $P_A v = v$. For this eigenvector v we can write $P_A v - P_B v = v - 0 = v$ which means that v is also an eigenvector of $P_A - P_B$ with eigenvalue 1.

  We have proven earlier that when P_A and P_B have same rank, then $P_A P_B^\perp$ and $P_B P_A^\perp$ they have the same number of eigenvalues with value 1. Using similar reasoning as above, eigenvector u of $P_B P_A^\perp$ with associated eigenvalue 1 is in the range of P_B and is orthogonal to the range of P_A. This would lead to $(P_A - P_B)u = P_A u - P_B u = 0 - u = -u$ which means that for each eigenvalue 1 of $P_B P_A^\perp$, $P_A - P_B$ has eigenvalue -1 associated with the same eigenvector u. Therefore, for any singular value 1 of both $P_A P_B^\perp$ and $P_B P_A^\perp$ we have ± 1 eigenvalues of $P_A - P_B$.

Now we need to prove that $P_A - P_B$ has eigenvalues ± 1 only when $P_A P_B^\perp$ and $P_B P_A^\perp$ have eigenvalues 1 to avoid cases where $P_A - P_B$ would have eigenvalues ± 1 without corresponding eigenvalues/singular values for $P_A P_B^\perp$ and $P_B P_A^\perp$.

Let us observe vector v that is an eigenvector of $P_A - P_B$ with eigenvalue 1. The following holds: $P_A v - P_B v = v \Rightarrow -P_B v = P_A^\perp v$. From there we have:

$$P_B^\perp v - v = P_A^\perp v \tag{38}$$

If we multiply both sides of the equation with $(P_A^\perp v)^*$ on the left side we get:

$$\begin{aligned}
&(P_A^\perp v)^* P_B^\perp v - (P_A^\perp v)^* v = (P_A^\perp v)^* P_A^\perp v \\
&-(P_B v)^* P_B^\perp v - v^* P_A^{\perp *} P_A^\perp v = (P_A^\perp v)^* P_A^\perp v \\
&-v^* P_B P_B^\perp v - (P_A^\perp v)^* P_A^\perp v = (P_A^\perp v)^* P_A^\perp v \\
&2 \cdot (P_A^\perp v)^* P_A^\perp v = 0 \Rightarrow P_A^\perp v = P_B v = 0
\end{aligned} \tag{39}$$

From the result in (39) we see that $P_A - P_B$ has eigenvalue 1 only for an eigenvector v that is in range of P_A and is orthogonal to the range of P_B. Such a vector is also an eigenvector of $P_A P_B^\perp$ with eigenvalue 1.

Using a similar approach we can prove that if a vector u is an eigenvector of $P_A - P_B$ with eigenvalue -1 then it is also an eigenvector of $P_B P_A^\perp$ with eigenvalue 1. From that and Lemma 1 we now know that for each eigenvalue pair ± 1 of $P_A - P_B$ there is a singular value 1 of both $P_A P_B^\perp$ and $P_B P_A^\perp$.

We have proven that nonzero singular values σ_i of $P_A P_B^\perp$ and $P_B P_A^\perp$ correspond to pairs of $\pm\sigma_i$ of eigenvalues of $P_B - P_A$. Since $P_B - P_A$ is a Hermitian matrix, absolute values of its eigenvalues are equal to its singular values (see [4], Theorem 5.5.) so we have:

$$\|P_A - P_B\|_2 = \|P_A P_B^\perp\|_2 = \|P_B P_A^\perp\|_2 \tag{40}$$

□

Proof. Proof of Theorem 2 We will prove this by proving the contrapositive. That is, that $rank(A) \neq rank(B) \Rightarrow \|P_B - P_A\|_2 \geq 1$.

Let us assume without loss of generality that $rank(A) > rank(B)$. We know that $dim(range(A)) = dim(range(P_A))$ and $dim(range(B)) = dim(range(P_B))$.

Let the dimension of column space of B be p and of column space of A be $p+k$ for some $k > 0$. We know that the dimension of the subspace orthogonal to the range of B is $m-p$ where m is the number of rows of matrices A and B. We have $dim(range(A)) + dim(range(P_B^{\perp})) = p+k+m-p = m+k$ with $k > 0$. This means that the intersection of range of A and the subspace orthogonal to the range of B is a non-empty set. If we pick a vector v from that intersection we get:

$$\begin{aligned} \|(P_B - P_A)v\|_2 = \|P_B v - P_A v\|_2 = \|0 - v\|_2 = \|v\|_2 \Rightarrow \\ \Rightarrow \|P_B - P_A\|_2 \geq \frac{\|(P_B - P_A)v\|_2}{\|v\|_2} = 1 \end{aligned} \tag{41}$$

We arrive to the same conclusion if we start with $rank(B) > rank(A)$.

□

References

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